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Free Energy of the Chiral Potts Model in the Scaling Region

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We explicitly calculate the free energy ψ of the general solvable N-state chiral Potts model in the scaling region, for $T < T_c$. We do this from both of the two available results for the free energy, and verify that they are mutually consistent. If $t = T_c - T$, then we find that $(\psi - \psi_c)/t$ has a Taylor expansion in powers of $t^{2/N}$ (together with higher-order non-scaling terms of order t, or $t \log t$).

KEY WORDS: Statistical mechanics; solvable lattice models; chiral Potts model.

1. INTRODUCTION

The free energy ψ_{pq} of the solvable chiral Potts model depends on four quantities: the number N of states per spin, a temperature-like parameter k', and explicitly on two rapidities p and q. It was first obtained in 1988,[1] yielding the critical exponent $\alpha = 1 - 2/N$. The method uses only the star–triangle relation for the model (ref. 2; ref. 3, pp. 83–87), showing that this implies partial differential equations for ψ_{pq}, involving a single-rapidity function G_p. However, the solution of these equations is intricate and far from transparent.

Alternative expressions as explicit integrals were obtained later[4] by solving the functional relations for the transfer matrices.[5] A fuller derivation is given in ref. 6, but regrettably there are inconsistencies in the choices of the variables v_p and v_q of Eqs. (52)–(64) therein: it seems that v_p and v_q should instead be chosen to lie between $-3\pi/2$ and $-\pi/2$, and that the result (64) is then correct for $-\pi < u_p < u_q < 0$. The results given in ref. 4, with $-\pi/2 < v_p$, $v_q < \pi/2$, are correct as written.

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It is by no means obvious that the solution of ref. 1 is the same as that of refs. 4 and 6. It would be interest to establish this directly, so as to better understand the analyticity properties of ψ_{pq}, and to obtain explicit expressions for the single-rapidity function G_p.

We have not yet succeeded in doing this, but here we do show that the two results lead to the same explicit result for ψ_{pq} in the scaling region near criticality. In fact we work not with ψ_{pq}, but with the quantity $\ln \tilde{\kappa}_{pq}$ related to it by (10) and (20): $\ln \tilde{\kappa}_{pq} = -\psi_{pq} - \ln(\rho_{pq} D_{pq})$.

More precisely, if k is the modulus of the model that is zero at criticality and unity at zero temperature, then near criticality k^2 is proportional to the temperature deviation $T - T_c$. The free energy has an expansion of the form

$$\ln \tilde{\kappa}_{pq} = P + Qk^2 + k^2 S(k^{4/N}) + O(k^4 \log k) \tag{1}$$

(Higher terms in the expansion are of the form $k^{2m+4n/N}$, possibly multiplied by $\log k$.) Here P, Q are independent of k, while $S(x)$ is a Taylor-expandable "scaling function," zero when x is zero. Here we evaluate P, Q, $S(x)$ from the integral expressions of refs. 4 and 6. They are the quantities $C_{pq}^{(1)}/4N$, $C_{pq}^{(2)}/4Nk^2$, $C_{pq}^{(3)}/4Nk^2$ of Section 4.

In Appendix A we check the equivalence of the various published forms for the critical free energy P (at which point the model reduces to the Fateev–Zamolodchikov model[7]). In Appendix B we verify that the method of ref. 1 gives the same results for Q, $S(x)$: this is an extension of the calculation in ref. 1, where we obtained P, Q and the first nonzero coefficient in the Taylor expansion of $S(x)$.

One interesting point is that both P and $S(x)$ (but not Q) depend on the vertical and horizontal rapidity variables u_p and u_q only via their difference $u_q - u_p$. In fact, $S(x)$ is simply proportional to $\sin(u_q - u_p)$. Thus, although the chiral Potts model does not in general have the rapidity difference property, we do regain it in the scaling region (provided we neglect terms analytic in k^2).

2. THE MODEL

We define the solvable chiral Potts model in the usual way.[2–6] Consider the square lattice of $\mathcal{N}$ sites and L columns, drawn diagonally as in Fig. 1, with toroidal (periodic) boundary conditions. At each site i there is a spin σ_i, which takes values $0, \ldots, N-1$. Adjacent spins interact with Boltzmann weights $W_{pq}(\sigma_i - \sigma_j)$ for SW → NE edges and $\overline{W}_{pq}(\sigma_i - \sigma_n)$ for SE → NW edges, as indicated.

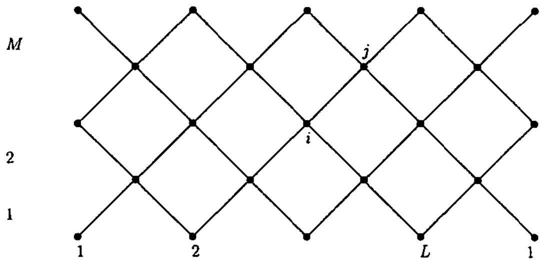

Fig. 1. The square lattice (drawn diagonally) with L columns and cylindrical boundary conditions.

We now define the functions $W_{pq}(n)$, $\bar{W}_{pq}(n)$. Let k be a real constant, $0 < k < 1$, $k' = (1 - k^2)^{1/2}$, and let $\omega = \exp(2\pi i/N)$. Let x_p, y_p, t_p, λ_p, μ_p, J_p be complex numbers ("p-variables"), related by

$$x_p^N + y_p^N = k(1 + x_p^N y_p^N), \qquad x_p y_p = t_p$$
$$k x_p^N = 1 - k' \lambda_p^{-1}, \qquad k y_p^N = 1 - k' \lambda_p \tag{2}$$
$$\lambda_p = \mu_p^N, \qquad J_p = -\lambda_p^2 x_p^N / y_p^N$$

We regard N and k as fixed parameters. Then if any one of the "p-variables" $x_p, \ldots, J_p$ is given, the rest are determined, to within a finite number of discrete choices of Nth roots and solutions of quadratic equations. In terms of the a_p, b_p, c_p, d_p of ref. 2, $x_p = a_p/d_p$, $y_p = b_p/c_p$, $\mu_d = d_p/c_p$, $J_p = -(a_p d_p / b_p c_p)^N$. We can regard the variables as being a point p on an algebraic curve (with one degree of freedom), and refer to this point as the "rapidity" p. The parameters t_p and λ_p are particularly significant: they are delated by

$$k^2 t_p^N = 1 - k'(\lambda_p + \lambda_p^{-1}) + k'^2 \tag{3}$$

As in ref. 1, we also introduce variables u_p, v_p related to one another and to those above by

$$\sin v_p = k \sin u_p, \qquad k'(\lambda_p - \lambda_p^{-1}) = 2k e^{iu_p} \cos v_p$$
$$x_p = e^{i(u - v)/N}, \qquad y_p = e^{i(\pi + u + v)/N}, \qquad t_p = e^{i(\pi + 2u)/N} \tag{4}$$
$$J_p = \frac{k'^2}{1 + k^2 - 2k \cos(u_p - v_p)} = \frac{\sin(u_p + v_p)}{\sin(u_p - v_p)}$$

Similarly, define "q-variables" x_q, y_q, t_q, λ_q, μ_q, J_q, u_q, v_q. Then the Boltzmann weights are, for all integers n,

$$
\begin{aligned}
W_{pq}(n) &= W_{pq}(0)\left(\frac{\mu_p}{\mu_q}\right)^n \prod_{j=1}^{n} \frac{y_q-\omega^j x_p}{y_p-\omega^j x_q} \\
\bar{W}_{pq}(n) &= \bar{W}_{pq}(0)(\mu_p\mu_q)^n \prod_{j=1}^{n} \frac{\omega x_p-\omega^j x_q}{y_q-\omega^j y_p}
\end{aligned}
\tag{5}
$$

In this paper we leave the normalization factors $W_{pq}(0)$, $\bar{W}_{pq}(0)$ arbitrary, except to require that they be real and positive, and have the rotation invariance property given below in Eq. (15).

They have the periodicity properties $W_{pq}(n+N)=W_{pq}(n)$, $\bar{W}_{pq}(n+N)=\bar{W}_{pq}(n)$. Here the rapidity p is associated with the vertical direction, q with the horizontal. We shall need the associated quantities

$$
\begin{aligned}
\rho_{pq} &= \left\{\prod_{n=0}^{N-1} W_{pq}(n)\right\}^{1/N}, & \bar{\rho}_{pq} &= \left\{\prod_{n=0}^{N-1} \bar{W}_{pq}(n)\right\}^{1/N} \\
D_{pq} &= \{\det_N[W_{pq}(i-j)]\}^{1/N}, & \bar{D}_{pq} &= \{\det_N[\bar{W}_{pq}(i-j)]\}^{1/N} \\
g_{pq} &= D_{pq}/\rho_{pq}, & \bar{g}_{pq} &= \bar{D}_{pq}/\bar{\rho}_{pq}
\end{aligned}
\tag{6}
$$

Explicit product formulas for $\bar{D}_{pq}$ are given in Eqs. (3.22) of ref. 1 and (2.44) of ref. 5. In (23) of ref. 6 these are put into the form

$$
\bar{g}_{pq} = N^{1/2}\eta^{-1/N}[(x_p^N-x_q^N)(y_p^N-y_q^N)]^{(1-N)/2N} \prod_{j=1}^{N-1} (t_p-\omega^j t_q)^{j/N} \tag{7}
$$

where

$$
\eta = e^{i\pi(N-1)N+4)/12} \tag{8}
$$

In Eq. (2.47) of ref. 5 it is remarked that

$$
g_{pq}\,\bar{g}_{pq} = Nk'^{(1-N)/N} \tag{9}
$$

The partition function depends on p and q, so we write it as Z_{pq}. Then the partition function and dimensionless free energy per site are

$$
\kappa_{pq} = Z_{pq}^{1/\mathcal{N}}, \qquad \psi_{pq} = -\ln \kappa_{pq} \tag{10}
$$

(In this notation, the $\psi_{pq}^{(Sq)}$ of Eq. (3.41) of ref. 1 and the ψ of Eq. (28) of ref. 6 are $\psi_{pq}+\ln[\rho_{pq}\bar{\rho}_{pq}]$; while the $V(t_q, \lambda_q)$ of ref. 4 is $\{\kappa_{pq}/(\rho_{pq}\bar{D}_{pq})\}^L$.)

2.1. Physical Regime

We can choose x_p, x_q, y_p, y_q, t_p, t_q so that they all lie on the unit circle, and are arranged so that

$$\arg(x_p) < \arg(x_q) < \arg(y_p) < \arg(y_q) < \arg(\omega x_p) \tag{11}$$

$$\arg(t_p) < \arg(t_q) < \arg(\omega t_p) \tag{12}$$

Using (2), the restrictions (11) imply (12); conversely, if t_p, t_q satisfy (12), there is a unique choice of x_p, x_q, y_p, y_q that satisfies (11). If $-2\pi/N < \arg(t_p) < 0$, then this choice ensures that $|\lambda_p| < 1$; if $0 < \arg(t_p) < 2\pi/N$, then $|\lambda_p| > 1$. Similarly for t_q and λ_q.

With these choices, all the Boltzmann weights $W_{pq}(n)$, $\bar{W}_{pq}(n)$ are real and positive, so the model is then physical: Z_{pq}, κ_{pq} must be real and positive; ψ_{pq} must be real. Here we shall focus our attention on this case, which we call the "physical regime." The parameters u_p, v_p, u_q, v_q are particularly useful in this regime. They are then real, satisfying

$$-\pi/2 < v_p < \pi/2, \qquad -\pi/2 < v_q < \pi/2, \qquad u_p < u_q < u_p + \pi \tag{13}$$

while J_p and J_q are real and positive.

Of course our results can be extended into the complex plane: such extensions can be very useful in any calculation, and vital in an understanding of the analyticity properties of κ_{pq}.

2.2. Rotation and Inversion Relations

An automorphism that plays a significant role in the model is $p \to Rp$, where

$$\begin{gathered} x_{Rp} = y_p, \qquad y_{Rp} = \omega x_p, \qquad \mu_{Rp} = 1/\mu_p \\ t_{Rp} = \omega t_p, \qquad u_{Rp} = u_p + \pi \end{gathered} \tag{14}$$

We require that the normalization factors $W_{pq}(0)$, $\bar{W}_{pq}(0)$ in (5) satisfy

$$W_{q,Rp}(0) = \bar{W}_{pq}(0), \qquad \bar{W}_{q,Rp}(0) = W_{pq}(0) \tag{15}$$

Then the weight functions and associated parameters have the properties (for all integers n, a, b)

$$\begin{gathered} W_{q,Rp}(n) = \bar{W}_{pq}(n), \qquad \bar{W}_{q,Rp}(n) = W_{pq}(-n) \\ \rho_{q,Rp} = \bar{\rho}_{pq}, \quad \bar{\rho}_{q,Rp} = \rho_{pq}, \qquad D_{q,Rp} = \bar{D}_{pq}, \quad \bar{D}_{q,Rp} = D_{pq} \end{gathered} \tag{16}$$

$$g_{q,Rp}=\bar{g}_{pq}, \qquad \bar{g}_{q,Rp}=g_{pq}, \qquad W_{pq}(n)\,W_{pq}(n)=\rho_{pq}\rho_{qp}$$

$$\sum_{c=0}^{N-1} \bar{W}_{pq}(a-c)\,\bar{W}_{qp}(c-b)=\bar{D}_{pq}\bar{D}_{pq} \qquad \text{if} \quad a=b, \quad \text{mod}\, N$$

$$=0 \qquad \text{otherwise} \tag{17}$$

The properties (16) ensure that replacing p, q by q, Rp is equivalent to rotation the lattice anticlockwise through 90°. This leaves the κ_{pq} and ψ_{pq} unchanged, so

$$\kappa_{q,Rp}=\kappa_{pq} \tag{18}$$

In the physical regime, it follows from (11) that x_p, x_q, y_p, y_q, ωx_p, ωx_q, ωx_q, ωy_p, ωy_q, $\omega^2 x_p, \ldots, \omega^{N-1} y_q$ form a set of $4N$ points ordered anticlockwise around the unit circle, the last element being following by the first. The mapping $p, q \rightarrow q, Rp$ simply replaces each element of this cyclically ordered set by the next. Hence κ_{pq} is unchanged if x_p, x_q, y_p, y_q are replaced by any other four consecutive elements of the set.

Further, the relations (17) imply the "inversion relation"[8]

$$\kappa_{pq}\kappa_{pq}=\rho_{pq}\rho_{qp}\bar{D}_{pq}\bar{D}_{pq} \tag{19}$$

where κ_{pq} is obtained by analytically continuing κ_{pq} through the inversion point $p=q$.

2.3. The Modified Partition Function per Site $\tilde{\kappa}_{pq}$

An associated quantity that we shall use is

$$\tilde{\kappa}_{pq}=\kappa_{pq}/(\rho_{pq}\bar{D}_{pq}) \tag{20}$$

[This is the $\exp(-\Lambda_{pq})$ of ref. 1 and the $V(t_q, \lambda_q)^{1/L}$ of ref. 4.] This is independent of the normalization factors $W_{pq}(0)$, $\bar{W}_{pq}(0)$. Using this, we find that the inversion relation (19) simplifies:

$$\tilde{\kappa}_{pq}\tilde{\kappa}_{pq}=1 \tag{21}$$

while the rotation symmetry (18) becomes more complicated:

$$\tilde{\kappa}_{q,Rp}=(\bar{g}_{pq}/g_{p,q})\,\tilde{\kappa}_{pq} \tag{22}$$

3. EXPRESSIONS FOR $\tilde{\kappa}_{pq}$

For $|\lambda_p|<1$, $|\lambda_q|<1$, and $-2\pi/N<\arg(t_q)<2\pi/N$, defined functions $\Delta(\theta)$, A_{pq}, B_{pq} by

$$\Delta(\theta)=[(1-2k'\cos\theta+k'^2)/k^2]^{1/N} \tag{23}$$

$$A_{pq}=(2\pi)^{-1}\int_0^{2\pi}\frac{1+\lambda_p e^{i\theta}}{1-\lambda_p e^{i\theta}}\sum_{j=1}^{N-1}(N-j)\ln[\Delta(\theta)-\omega^j t_q]\,d\theta \tag{24}$$

$$B_{pq}=(8\pi^2)^{-1}\int_0^{2\pi}\int_0^{2\pi}\frac{1+\lambda_p e^{i\theta}}{1-\lambda_p e^{i\theta}}\frac{1+\lambda_q e^{i\phi}}{1-\lambda_q e^{i\phi}}$$
$$\times\sum_{j=1}^{N-1}(N-2j)\ln[\omega^{-j/2}\Delta(\theta)-\omega^{j/2}\Delta(\phi)]\,d\theta\,d\phi \tag{25}$$

Then $B_{qp}=-B_{pq}$ and in ref. 6 we show that

$$N\ln\tilde{\kappa}_{pq}=[(N-1)/2]\ln(\lambda_q/\lambda_p)+A_{pq}-A_{qp}-B_{pq} \tag{26}$$

provided $|\lambda_p|<1$, $|\lambda_q|<1$, $-2\pi/N\arg t_p<0$, and $-2\pi/N\arg t_q<0$.

We can write these integrals in various ways, some of which manifest the fact that $\tilde{\kappa}_{pq}$ is real in the physical regime. In particular, if we introduce the Fourier transform funtion

$$G_p(\beta)=-\frac{\cos v_p}{\pi}\int_{-\infty}^{\infty}\frac{\exp[\beta+2\beta(u_p+ix)/\pi]\,dx}{\sin(u_p+ix)(1+k^2\sinh^2 x)^{1/2}} \tag{27}$$

then in ref. 6 it is shown that

$$4N\ln\tilde{\kappa}_{pq}=(N-1)\ln\left(\frac{J_q}{J_p}\right)+\mathrm{P}\int_{-\infty}^{\infty}\frac{E_{pq}(\beta)\exp[2\beta(u_q-u_p)/\pi]\,d\beta}{\beta\sinh N\beta} \tag{28}$$

where P indicates the principal-value integral and

$$E_{pq}(\beta)=[G_p(\beta)\,G_q(-\beta)+\operatorname{cosech}^2(\beta)]$$
$$\times[N\sinh\beta\cosh(N-1)\beta-\sinh N\beta]$$
$$+N\sinh(N-1)\beta[G_p(\beta)+G_q(-\beta)] \tag{29}$$

provided both u_p, u_q lie in the interval $(-\pi,0)$, and v_p, v_q in the interval $(-\pi/2,0)$. [There is some confusion in Eqs. (52)–(65) of ref. 6 as to the choice of v_p, v_q: if we choose them as we do here, then the definition of $G_p(\beta)$ in Eq. (55) of ref. 6 has to be negated, giving (27). The result reported in ref. 4 is correct as written.]

We can extend these results for $\tilde{\kappa}_{pq}$ to the remainder of the physical regime, either by analytic continuation (taking care to form the correct continuation when, for instance, a pole crosses a contour of integration), or more easily by using the rotation symmetry (18), (22). Boundary cases can be handled by taking an appropriate limit.

It is readily seen (by negating β) that the right-hand sides of (26) and (28) are antisymmetric functions of p and q, in argument with (1). Furthermore, it has recently been verified explicitly that the analytic continuation of (26) does indeed satisfy the rotation symmetry.[9]

4. THE SCALING REGION

At $k=0$ the model becomes the critical Fateev–Zamolodchikov model.[7] Here we are interested in the behavior as this critical limit is approached. One can verify that the Boltzmann weights $W_{pq}(n)$, $\bar{W}_{pq}(n)$ are even functions of k, expandable in powers of k^2, so k^2 plays the role of the temperature deviation from criticality $T_c - T$.

At least for N even, some of the neglected terms in the expansion also contain a factor $\log k$. To avoid irritating repetition, if we say that we are neglecting terms of order k^n, then we are also neglecting terms of order $k^n \log k$.

Let

$$z_0 = \frac{1-k'}{1+k'} = \frac{k^2}{(1+k')^2} = \exp\left[-2 \operatorname{arcosh}\left(\frac{1}{k}\right)\right] \tag{30}$$

Then by integrating the integrand in (27) around the rectangle with vertices $-S$, S, $S+i\pi$, $-S+i\pi$, allowing for branch cuts from $i\pi/2 \pm \operatorname{arcosh}(1/k)$ to $i\pi/2 \pm \infty$ and the pole at $i(\pi+u_p)$, and letting $S \to \infty$, we can rewrite (7) as

$$G_p(\beta) = [1 + H_p(\beta)]/\cosh\beta \tag{31}$$

where

$$H_p(\beta) = -\frac{i \cos v_p}{\pi} e^{\beta(1+2u_p/\pi)}[V_p(\beta) - V_p^*(\beta)] \tag{32}$$

$$V_p(\beta) = 2k^{-1} z_0^{\frac{1}{2} - i\beta/\pi} e^{iu_p} \int_0^1 \frac{t^{-\beta/\pi}\, dt}{(1+e^{2iu_p} z_0 t)[(1-t)(1-z_0^2 t)]^{1/2}} \tag{33}$$

and $V_p^*(\beta)$ is defined similarly, but with i replaced by $-i$. [Thus it is the complex conjugate of $V_p(\beta)$ if k, z_0, u_p, and β are real.] $V_p(\beta)$ and

$V_q^*(-\beta)$ are bounded analytic functions of β on the real axis and in the UHP. One can verify by direct integration that

$$2\cos v_p\, V_p(0) = 2iv_p + \ln J_p \tag{34}$$

Similarly, $2\cos v_p\, V_p^*(0) = -2iv_p + \ln J_p$, and hence $G_p(0) = 1 + 2v_p/\pi$.

Substituting (31) into (29), we obtain

$$E_{pq}(\beta) = \sum_{j=1}^{3} E_{pq}^{(j)}(\beta) \tag{35}$$

where

$$\begin{aligned}
E_{pq}^{(1)}(\beta) &= \frac{N\sinh\beta\cosh(N+1)\,\beta - \sinh N\beta\cosh 2\beta}{\sinh^2\beta\cosh^2\beta} \\
E_{pq}^{(2)}(\beta) &= \frac{(N-1)\sinh N\beta[H_p(\beta) + H_q(-\beta)]}{\cosh^2\beta} \\
E_{pq}^{(3)}(\beta) &= \frac{[N\sinh\beta\cosh(N-1)\beta - \sinh N\beta]\, H_p(\beta)\, H_q(-\beta)}{\cosh^2\beta}
\end{aligned} \tag{36}$$

As $k \to 0$, z_0 also tends to zero (to leading order it is $k^2/4$), so $V_p^*(\beta)$, $V_q(-\beta)$, $V_q^*(-\beta)$, $H_p(\beta)$, $H_q(-\beta)$ all become small. The equations are therefore in a form where we can examine the critical behavior. To do this, it is convenient to consider separately the contributions to the RHS of (28) of the terms $E^{(1)}$, $E^{(2)}$, $E^{(3)}$.

4.1. Contribution from $E^{(1)}$

The term $E^{(1)}$ in (36) gives a contribution to (28) of

$$C_{pq}^{(1)} = \mathrm{P}\int_{-\infty}^{\infty} \frac{E_{pq}^{(1)}(\beta)\exp[2\beta(u_q - u_p)/\pi]\, d\beta}{\beta\sinh N\beta} \tag{37}$$

This is independent of k and is the only nonzero contribution in the limit $k \to 0$. (J_p and J_q both tend to 1.) It is therefore the free energy $4N\ln\tilde{\kappa}_{pq}$ of the Fateev–Zamolodchikov model[7] [ref. 1, Eqs. (6.11) and (6.15)].

We now consider the other contributions to (28).

4.2. Contributions from $(N-1)\ln(J_q/J_p)$ and $E^{(2)}$

The terms arising from $E^{(2)}$ are linear in the V's and each corresponding integral can be closed around either the upper or lower half-plane.

Because $E^{(2)}$ contains a factor $\sinh\beta$, the only singularities are single poles at $\beta=0$ and double poles at $\beta=i(2n-1)\pi/2$ (n an integer). The poles at $\beta=0$ given a combined contribution to (28) of

$$(N-1)\cos v_p[V_p(0)+V_p^*(0)-V_q(0)-V_q^*(0)] \tag{38}$$

Using (34), this is $-(N-1)\ln(J_q/J_p)$. Thus the contribution of the poles at $\beta=0$ precisely cancels the $(N-1)\ln(J_q/J_p)$ term in (28). This ensures that there are no terms of order k (or of higher odd powers of k) in the expansion of (28).

The next highest order contribution comes from the poles at $\beta=\pm i\pi/2$ in $E^{(2)}$, and is of order k^2 (or, because the poles are double, $k^2\log k$). To this order we can set $z_0=0$ inside the integrand of (33), giving

$$V_p(\beta)=2k^{-1}z_0^{1-i\beta/\pi}e^{iu_p}B(1-i\beta/\pi, 1/2) \tag{39}$$

where $B(x,y)=\Gamma(x)\Gamma(y)/(x+y)$ is Euler's beta function.

We can also take $\cos v_p$ in (32) to be unity, giving a contribution to (28) of

$$S_{pq}+S_{pq}^*-S_{pq}-S_{pq}^* \tag{40}$$

where

$$S_{pq}=\frac{2(N-1)}{\pi ik}e^{iu_p}\,\mathrm{P}\int_{-\infty}^{\infty}\frac{z_0^{1-i\beta/\pi}e^{\beta(1+2u_q/\pi)}}{\beta\cosh^2\beta}B\left(1-\frac{i\beta}{\pi},\frac{1}{2}\right)d\beta \tag{41}$$

and S_{pq}^* is the "complex conjugate" obtained from it by replacing i by $-i$, wile leaving k, z_0, u_p, u_q, β unchanged.

The function $B(1-i\beta/\pi, 1/2)$ is bounded and analytic in the upper half β plane, so the integral can be closed round the UHP, and the contribution in which we are interested comes from the double pole at $\beta=i\pi/2$. We can break the residue at this pole into two parts:

(a) The part coming from the first derivative at $\beta=i\pi/2$ of the factor $\beta^{-1}z_0^{1-i\beta/\pi}e^{\beta}B(1-i\beta/\pi, 1/2)$. This depends on the rapidities p and q only via a factor $\exp[i(u_p+u_q)]$. It is therefore symmetric in p and q, so cancels out of (40) and can be ignored. For this reason there are no $k^2\log k$ terms in the contribution.

(b) The part coming from the first derivative of $\exp(2\beta u_q/\pi)$. To leading order (setting $z_0=k^2/4$), this contributes of S_{pq} a term

$$-2(N-1)k^2\pi^{-2}u_q e^{i(u_p+u_q)}B(3/2, 1/2) \tag{42}$$

Noting that $B(3/2, 1/2) = \pi/2$, it follows that the contribution to (28) of the terms of order k^2 coming from $E^{(2)}$ is

$$C_{pq}^{(2)} = -2(N-1)\, k^2 \pi^{-1} (u_q - u_p) \cos(u_p + u_q) \tag{43}$$

There are other terms coming from the expansion of the integrand in (33) in powers of z_0, and from the poles at $\beta = 3i\pi/2$, $5i\pi/2$, etc. These are of order $k^4, k^6, \ldots$.

4.3. Contributions from $E^{(3)}$

Substituting the forms (32) of $H_p(\beta)$, $H_q(-\beta)$ into (36), we can break $E_{pq}^{(3)}(\beta)$ into two parts, one containing $V_p(\beta)\, V_q(-\beta)$ and $V_p^*(\beta)\, V_q^*(-\beta)$, the other containing $V_p(\beta)\, V_q^*(-\beta)$ and $V_p^*(\beta)\, V_q(-\beta)$.

The first involves k and z_0 only via an external factor z_0^2/k^2, and the z_0 inside the integrand in (33). The corresponding contribution to (28) can therefore be expanded in powers of k^2. At first sight there is a leading term of order k^2, but it is an integral over β from $-\infty$ to ∞ of an odd function of β, so it vanishes. The surviving terms are at most of order k^4.

Now consider the term containing $V_p(\beta)\, V_q^*(-\beta)$. Ignoring terms of relative order k^2, we can replace $H_p(\beta)\, H_q(-\beta)$ in Eq. (36) for $E^{(3)}$ by $\exp[2\beta(u_p - u_q)/\pi]\, V_p(\beta)\, V_q^*(-\beta)/\pi^2$, i.e. [using (39)] by

$$(2/\pi k)^2 z_0^{2-2i\beta/\pi} e^{2\beta(u_p - u_q)/\pi} e^{i(u_p - u_q)} B(1 - i\beta/\pi, 1/2)^2 \tag{44}$$

The resulting contribution to the integral in (28) can be closed around the UHP, the integrand being analytic except for poles arising from the factors $\sinh N\beta$, $\cosh^2 \beta$ in the denominator.

The factor $z_0^{2-2i\beta/\pi}$ ensures that (for k small) the dominant contribution to the integral comes from the poles closest to the origin, i.e., $\beta = i\pi j/N$ for $j = 1, 2, \ldots$, where $\sinh N\beta$ vanishes. (There is no pole at the origin.) Noting that $\sinh N\beta$ is then zero, we can replace the definition (36) of $E^{(3)}$ by $N \tanh \beta \cosh N\beta H_p(\beta)\, H_q(-\beta)$. The pole at $\beta = i\pi j/N$ therefore contributes to (28) a term

$$\frac{8Ni}{\pi^2 j k^2} \tan\left(\frac{\pi j}{N}\right) z_0^{2+2j/N} e^{i(u_p - u_q)} B\left(1 + \frac{j}{N}, \frac{1}{2}\right)^2 \tag{45}$$

The term containing $V_p^*(\beta)\, V_q(-\beta)$ is the "complex conjugate" of this, so altogether (again taking $z_0 = k^2/4$) we obtain a contribution to (28) of

$$C_{pq}^{(3)} = \frac{Nk^2}{\pi^2} \sin(u_q - u_p) \sum_{j=1}^{\infty} j^{-1} \left(\frac{k}{2}\right)^{4j/N} \tan\left(\frac{\pi j}{N}\right) B\left(1 + \frac{j}{N}, \frac{1}{2}\right)^2 \tag{46}$$

If N is even, a problem arises when j is an odd multiple of $N/2$ (due to the integrand having a double pole). However, we are neglecting terms of order k^4, so should restrict the sum in (46) to $1 \leqslant j < N/2$, which removes the difficulty.

Ignoring terms of order k^4 or smaller, we thus have

$$4N \ln \tilde{\kappa}_{pq} = C_{pq}^{(1)} + C_{pq}^{(2)} + C_{pq}^{(3)} \tag{47}$$

As discussed in the introduction, $C_{pq}^{(1)}$ is the contribution of the critical free energy (i.e., the Fateev–Zamolodchikov model) and is given by (37); $C_{pq}^{(2)}$ is the first analytic correction, being proportional to k^2 and given by (43); $C_{pq}^{(3)}$ is the scaling contribution, given by (46). Note that $C_{pq}^{(3)}$, consider as a function of k and the rapidity variables u_p and u_q, has the form

$$C_{pq}^{(3)} = k^2 \sin(u_q - u_p)\, F(k^{4/N}) \tag{48}$$

where $F(x)$ is a Taylor-expandable function of x. [If we truncate the series in (47) to $1 \leqslant j < N/2$, as remarked above, then it is a polynomial.]

APPENDIX A

Here we consider the critical $k \to 0$ limit, when the model reduces to that of Fateev and Zamolodchikov, and show that our expressions (28), (37) then agree with previous results in refs. 7 and 1.

From (2)–(4) and (13), in this limit $v_p = v_q = 0$ and we can choose $\mu_p = \mu_q = 1$. Then (5) gives

$$\begin{aligned} W_{pq}(n) &= W_{pq}(0) \prod_{j=1}^{n} \frac{\sin[\pi j/N - (\pi + \alpha)/2N]}{\sin[\pi j/N - (\pi - \alpha)/2N]} \\ \bar{W}_{pq}(n) &= \bar{W}_{pq}(0) \prod_{j=1}^{n} \frac{\sin[\pi(j-1)/N + \alpha/2N]}{\sin[\pi j/N - \alpha/2N]} \end{aligned} \tag{A1}$$

where $\alpha = u_q - u_p$. These formulas agree (to within a normalization) with those of Fateev and Zamolodchikov.[7] Also (6), (7) give

$$\bar{D}_{pq}/\bar{\rho}_{pq} = N^{1/2} (2 \sin \alpha/2)^{(1-N)/N} \prod_{j=1}^{N-1} [2 \sin(\alpha + \pi j)/N]^{j/N} \tag{A2}$$

For $0 < a$, $b < \pi$, one has the formula

$$\ln \left[\frac{\sin a}{\sin b} \right] = \mathrm{P} \int_{-\infty}^{\infty} \frac{e^{-t}(e^{2bt/\pi} - e^{2at/\pi})\, dt}{2t \sinh t}$$

from which one can deduce, for $0<\alpha<\pi$, that

$$
\begin{aligned}
4N\ln\left[\frac{\rho_{pq}}{W_{pq}(0)}\right] &= \mathrm{P}\int_{-\infty}^{\infty}\frac{e^{2\alpha\beta/\pi}}{\beta\sinh N\beta}\,g_1(\beta)\,d\beta \\
4N\ln\left[\frac{\bar{D}_{pq}}{\bar{W}_{pq}(0)}\right] &= 2N\ln N+\mathrm{P}\int_{-\infty}^{\infty}\frac{e^{2\alpha\beta/\pi}}{\beta\sinh N\beta}\,g_2(\beta)\,d\beta
\end{aligned}
\tag{A3}
$$

where

$$
\begin{aligned}
g_1(\beta) &= \frac{2\cosh 2\beta\sinh N\beta}{\sinh^2 2\beta}-\frac{N\cosh 2N\beta}{\cosh N\beta\sinh 2\beta} \\
g_2(\beta) &= \frac{Ne^{2(N-1)\beta}}{\sinh 2\beta\cosh N\beta}+\frac{2\sinh N\beta\cosh 2\beta}{\sinh^2 2\beta}-\frac{Ne^{(N-1)\beta}}{\sinh\beta}
\end{aligned}
\tag{A4}
$$

From (20), (28), and (37), for $k=0$ our result is

$$
4N\ln\left[\frac{\kappa_{pq}}{\rho_{pq}\bar{D}_{pq}}\right]=\mathrm{P}\int_{-\infty}^{\infty}\frac{e^{2\alpha\beta/\pi}}{\beta\sinh N\beta}\,E_{pq}^{(1)}(\beta)\,d\beta
$$

Using (A3), we can write this written as

$$
\ln\left[\frac{\kappa_{pq}}{W_{pq}(0)\,\bar{W}_{pq}(0)}\right]=\frac{1}{2}\ln N+\int_{-\infty}^{\infty}\frac{e^{2\alpha\beta/\pi}}{4N\beta\sinh N\beta}\,h(\beta)\,d\beta \tag{A5}
$$

where

$$
\begin{aligned}
h(\beta) &= E_{pq}^{(1)}(\beta)+g_1(\beta)+g_2(\beta) \\
&= -Ne^{-\beta}\frac{\sinh(N-1)\beta\sinh N\beta}{\cosh^2\beta\cosh N\beta}
\end{aligned}
\tag{A6}
$$

The variables u_p, u_q in ref. 1 are the same as those here. Also, Λ_{pq}, $W_{pq}(1,1)$, $\bar{W}_{pq}(1,1)$, $f(u_q-u_p)$ therein are our expressions $-\ln\bar{\kappa}_{pq}$, $W_{pq}(0)/\rho_{pq}$, $\bar{W}_{pq}(0)/\bar{\rho}_{pq}$, $\bar{D}_{pq}/\bar{\rho}_{pq}$. Thus in our present notation the result (6.11), (6.15) of ref. 1 (for $k=0$) is

$$
\ln\left[\frac{\kappa_{pq}}{W_{pq}(0)\,\bar{W}_{pq}(0)}\right]=\int_0^{\infty}\frac{\sinh\alpha x\sinh(\pi-\alpha)x\sinh(N-1)\pi x}{x\cosh^2\pi x\cosh N\pi x}\,dx \tag{A7}
$$

Setting $x=\beta/\pi$, we can write this as

$$
\ln\left[\frac{\kappa_{pq}}{W_{pq}(0)\,\bar{W}_{pq}(0)}\right]=\int_{-\infty}^{\infty}\frac{[\cosh\beta-\cosh(2\alpha/\pi-1)\beta]\sinh(N-1)\beta}{4\beta\cosh^2\beta\cosh N\beta}\,d\beta \tag{A8}
$$

Noting the evenness of the integrand and that

$$\int_{-\infty}^{\infty} \frac{\sinh(N-1)\beta}{\beta \cosh \beta \cosh N\beta}\, d\beta = 2 \ln N$$

we see that this is the same as our result (A5).

It is also the same as Eq. (12) of ref. 7, provided we normalize so that $W_{pq}(0) = \bar{W}_{pq}(0) = 1$. [In fact it appears from Eq. (2) of ref. 7 that Fateev and Zamolodchikov normalized $\xi_{pq} = \sum_{n=0}^{N-1} W_{pq}(n)/N$ and $\bar{\xi}_{pq} = \sum_{n=0}^{N-1} \bar{W}_{pq}(n)/N$ to be unity, which means that their "specific" free energy should be $-\ln(\kappa_{pq}/\xi_{pq}\bar{\xi}_{pq})$. Since $W_{pq}(0)\, \bar{W}_{pq}(0)/\xi_{pq}\bar{\xi}_{pq} = N$, this implies that the RHS of Eq. (12) of ref. 7 should contain an additional term $-\ln N$.]

APPENDIX B

Here we use the alternative method of ref. 1 to evaluate the free energy to the same order as in (47). In particular we rederive the contributions $C_{pq}^{(2)}$ and $C_{pq}^{(3)}$. (This method does not immediately give $C_{pq}^{(1)}$, which plays the role of an undetermined constant of integration, independent of k and depending on u_p and u_q only via their difference $u_q - u_p$.) The results agree (as of course they should) with (43) and (46), and are consistent with (37).

We denote the equations of ref. 1 by the prefix I. The summand in (I.5.37) is unchanged by $j \to N-j$ and when k is small the integral is dominated by the region where l is of order k, hence

$$\lambda(k) = -\frac{8}{\pi^3 N^2} \sum_{j=1}^{N-1} (N-2j) \sin^2\left(\frac{\pi j}{N}\right) \times \int_0^{\infty} \frac{2}{(k^2+l^2)^2} K^2_{(2j-N)/2N}(l)\, l\, dl \tag{B1}$$

When k is small, from (I.5.7),

$$K_n(k) = \tfrac{1}{2} k^{2n} B(n+1/2, n+1/2)$$

while from (3.194.6) of ref. 10,

$$\int_0^{\infty} \frac{l^{(4j-N)/N}\, dl}{(k^2+l^2)^2} = k^{4(j-N)/N} \frac{(N-2j)\pi}{2N \sin(2\pi j/N)}$$

so (B1) becomes

$$\lambda(k) = -\sum_{j=1}^{N-1} \gamma_j k^{4(j-N)/N} \tag{B2}$$

where

$$\gamma_j = [(N-2j)^2/\pi^2 N^3] \tan(\pi j/N)\, B(j/N, j/N)^2 \tag{B3}$$

From (I.5.6), neglecting terms of relative order k^2, it follows that

$$\hat{G}_n(k) = B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) \sum_{j=1}^{N-1} \frac{N^2 \gamma_j k^{2n-2+4j/N}}{8(N-2j)(2j-N+2nN)} \tag{B4}$$

From (I.5.5), (I.4.13), (I.4.12), and (I.6.9), it follows that, to this order in each Fourier coefficient, the functions x_p, y_p of ref. 1 are

$$\begin{aligned} x_p &= -H_0 - 2 \sum_{n=1}^{\infty} H_n(k) \cos 2nu_p \\ y_p &= \frac{N-1}{N\pi} k^2 u_p + 2 \sum_{n=1}^{\infty} H_n(k) \sin 2nu_p \end{aligned} \tag{B5}$$

where

$$H_n(k) = (-1)^n \sum_{j=1}^{N-1} \frac{N^2 \gamma_j k^{2n+4j/N}}{2^{n+2}(N-2j)^2} \prod_{m=0}^{n-1} \frac{j+mN}{2j+N+2mN} \tag{B6}$$

Substituting these results into (I.3.45) and (I.3.46) and ignoring (for given u_p and u_q) contributions to Λ_{pq} of order k^4 or smaller, we need only retain the coefficient $H_1(k)$ and the term in y_p linear in u_p, giving

$$-4N\Lambda_{pq} = 4N \ln \tilde{\kappa}_{pq} = D_{pq}^{(1)} + D_{pq}^{(2)} + D_{pq}^{(3)} \tag{B7}$$

where $D_{pq}^{(1)}$ is a function of $u_q - u_p$ only, independent of k, but is otherwise at this stage undetermined, and

$$D_{pq}^{(2)} = -(2/\pi)(N-1)\, k^2(u_q - u_p) \cos(u_p + u_q) \tag{B8}$$

$$\begin{aligned} D_{pq}^{(3)} &= -16N \sin(u_q - u_p) \int_0^k l^{-1} H_1(l)\, dl \\ &= N^4 \sin(u_q - u_p) \sum_{j=1}^{N-1} j\gamma_j k^{2+4j/N}/(N^2-4j^2)^2 \end{aligned} \tag{B9}$$

Noting (using formula 8.335 of ref. 10) that

$$B(x, x) = (2x+1)\, B(1+x, 1/2)/(4^x x) \tag{B10}$$

it follows that

$$D_{pq}^{(3)} = (Nk^2/\pi^2) \sin(u_q - u_p) \sum_{j=1}^{N-1} j^{-1}(k/2)^{4j/N} \tan(\pi j/N)\, B(1 + j/N, 1/2)^2 \tag{B11}$$

If we truncate the sum in (B9) to only the $j=1$ term, then we regain the result (I.6.11).

We want to assert that the main result (47) of this paper is consistent with the result (B7) of this appendix, more strongly that $C_{pq}^{(i)} = D_{pq}^{(i)}$ for $i = 1, 2, 3$. Certainly $C_{pq}^{(1)}$ is a function of $u_q - u_p$ only, independent of k, and so has the form allowed for $D_{pq}^{(1)}$. Also, from (43) with (B8), $C_{pq}^{(2)} = D_{pq}^{(2)}$. As written, the sums in (46) and (B11) have different upper limits, but, as remarked after (46), to order less than k^4 both should be restricted to the range $1 \leqslant j < N/2$. (This restriction also removes the problem that many of the summands of this appendix are undetermined or infinite for $j = N/2$.) Then we obtain $C_{pq}^{(3)} = D_{pq}^{(3)}$. Thus (7) is consistent with (47).

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